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1995 J. Phys. A: Math. Gen. 28 L245

(http://iopscience.iop.org/0305-4470/28/8/004)

J. Phys. A: Math. Gen. 28 (1995) L245–L248. Printed in the UK

LETTER TO THE EDITOR

Supersymmetric Harry Dym type equations

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Received 9 January 1995

Abstract. A supersymmetric version is proposed for the well known Harry Dym system. A general-class super Lax operator which leads to consistent equations is considered.

During the past ten years or so, super extensions of integrable models have been subjected to much attention. The consequence of such study is that a number of well known integrable systems have been embedded in the context of super systems. In particular, we mention here the super Sine–Gordon [1], Korteweg–de Vries (KdV) [2–4] and nonlinear Schrödinger equations [5] and super Kadomtsev–Petviashvili hierarchy [2], etc (see [6] for more references).

We note that two types of super extensions for a given integrable system may exist, i.e. supersymmetric and fermionic extensions. In the KdV case, this corresponds to Manin–Radul's version [2] and Kupershmidt's version [4] respectively. Apart from the KdV system, the Harry Dym (HD) equation is also well known. Very recently, it has been found that the HD equation is not just a mathematically interesting model as it possesses physical applications [7]. Thus, it would be interesting to construct a super analogy of the HD equation. In this regard, a fermionic HD model is known from Kupershmidt's work [8] while a generic supersymmetric HD (sHD) system is still lacking to the best of my knowledge. The aim of this letter is to propose such a model.

For convenience, we fix our notation at the very beginning: denoting even variables by Latin letters and odd variables by Greek letters; the index $\geqslant_r$ of an operator always means projection to that part of the order that is greater than D^r (including the term D^r itself).

Let us first recall some basic facts of the HD equation. The equation reads

$$w_t = \tfrac{1}{4} w^3 w_{xxx} \tag{1}$$

or

$$u_t = \tfrac{1}{4}(u^{\frac{3}{2}} u_{xxx} - \tfrac{3}{2} u^{\frac{1}{2}} u_x u_{xx} + \tfrac{3}{4} u^{-\frac{1}{2}} u_x^3). \tag{2}$$

The link between them is $u = w^2$.

It is known that HD equation (2) has the following Lax operator:

$$L_{\mathrm{HD}} = u\partial^2 \tag{3}$$

and the Lax equation is

$$L_{\mathrm{HD}_t} = [P, L_{\mathrm{HD}}] \tag{4}$$

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where $P = (L^{\frac{3}{2}})_{\geqslant 4}$.

Our candidate for the supersymmetric Harry Dym(sHD) equation is:

$$\begin{aligned}
u_t = {} & (\tfrac{1}{4}u^{\frac{3}{2}}u_{xxx} - \tfrac{3}{8}u^{\frac{1}{2}}u_x u_{xx} + \tfrac{3}{16}u^{-\frac{1}{2}}u_x^3) + \tfrac{3}{8}u^{\frac{1}{2}}\alpha(Du_{xx}) - \tfrac{3}{4}u^{\frac{1}{2}}\alpha\alpha_{xx} - \tfrac{3}{32}u^{-\frac{3}{2}}u_x^2\alpha(Du) \\
& + \tfrac{3}{16}u^{-\frac{1}{2}}u_{xx}\alpha(Du) + \tfrac{3}{8}u^{-\frac{1}{2}}u_x\alpha_x(Du) - \tfrac{3}{8}u^{-\frac{1}{2}}u_x\alpha(Du_x) - \tfrac{3}{8}u^{\frac{1}{2}}\alpha_{xx}(Du) \\
& + \tfrac{3}{16}u^{-\frac{3}{2}}u_x\alpha(D\alpha)(Du) - \tfrac{3}{8}u^{-\frac{1}{2}}\alpha(D\alpha_x)(Du) + \tfrac{3}{4}u^{-\frac{1}{2}}\alpha\alpha_x u_x \\
\alpha_t = {} & \tfrac{1}{4}u^{\frac{3}{2}}\alpha_{xxx} + \tfrac{3}{8}u^{\frac{1}{2}}\alpha(D\alpha_{xx}) + \tfrac{3}{16}u^{-\frac{1}{2}}u_x^2\alpha_x - \tfrac{3}{8}u^{\frac{1}{2}}u_{xx}\alpha_x - \tfrac{9}{32}u^{-\frac{3}{2}}u_x^2\alpha(D\alpha) \\
& + \tfrac{3}{16}u^{-\frac{1}{2}}u_{xx}\alpha(D\alpha) + \tfrac{3}{8}u^{-\frac{1}{2}}u_x\alpha_x(D\alpha) - \tfrac{3}{8}u^{\frac{1}{2}}\alpha_{xx}(D\alpha) \\
& + \tfrac{3}{16}u^{-\frac{3}{2}}u_x\alpha(Du)\alpha_x - \tfrac{3}{8}u^{-\frac{1}{2}}\alpha(Du_x)\alpha_x + \tfrac{3}{16}u^{-\frac{3}{2}}u_x\alpha(D\alpha)^2 \\
& + \tfrac{3}{16}u^{-\frac{3}{2}}\alpha(Du)\alpha_x(D\alpha) - \tfrac{3}{8}u^{-\frac{1}{2}}\alpha(D\alpha)(D\alpha_x)
\end{aligned} \tag{5}$$

where $D = \theta\partial + \partial_\theta$, u is a super even variable and α is a super odd variable. Since the system is formulated in superderivative and super fields, the supersymmetry is manifest. If we set the odd field variable α to zero, we obtain the HD equation (2), thus system (5) deserves the name of sHD.

System (5) has the following Lax representation:

$$L_t = [P, L] \tag{6}$$

where $L = u\partial^2 + \alpha\partial D$, and

$$\begin{aligned}
P = (L^{\frac{3}{2}})_{\geqslant 3} = {} & u^{\frac{3}{2}}\partial^3 + \tfrac{3}{2}u^{\frac{1}{2}}\alpha\partial^2 D + \left(\tfrac{3}{4}u^{\frac{1}{2}}u_x + \tfrac{3}{8}u^{-\frac{1}{2}}\alpha(Du)\right)\partial^2 \\
& + \left(\tfrac{3}{4}u^{\frac{1}{2}}\alpha_x + \tfrac{3}{8}u^{-\frac{1}{2}}\alpha(D\alpha)\right)\partial D.
\end{aligned}$$

Noticing that the sHD system can be reformulated in Lax form, we see that this type of representation is non-standard in the Kupershmidt sense [9] (see also Kiso [10]). A detailed presentation of the non-standard Lax representation can be found in [11]. This prompted us to consider the more general operator:

$$L = u\partial^2 + \alpha\partial D + v\partial + \beta D + w. \tag{7}$$

Taking L as a Lax operator, we may construct integrable systems by means of the fractional power method. It is not difficult to verify that the following four cases occur.

Case 1:

$$L_t = [(L^{\frac{3}{2}})_{\geqslant 0}, L]. \tag{8}$$

The standard argument shows that the system is consistent: since $[(L^{\frac{3}{2}})_{\geqslant 0}, L] = -[(L^{\frac{3}{2}})_{<0}, L]$, the right-hand side of (8) is a differential operator of form $A\partial + \gamma D + B$. Thus, we may set $u = 1, \alpha = 0$. However, this implies $v = 0$ and we have the Manin–Radul case [2, 3].

Case 2:

$$L_t = [(L^{\frac{3}{2}})_{\geqslant 1}, L]. \tag{9}$$

The same argument leads to the conclusion: we may set $u = 1, \alpha = 0$. Thus, here we have a system of three equations. It is easy to see that we may further set $w = 0$. This last case was noticed by Inami and Kanno [6].

Case 3:

$$L_t = [(L^{\frac{3}{2}})_{\geqslant 2}, L]. \tag{10}$$

The general case will lead to a system involving five fields. However, we may have a subsystem which only has three fields.

Case 4:

$$L_t = [(L^{\frac{3}{2}})_{\geqslant 3}, L]. \tag{11}$$

As above, this system involves all five fields. A reduction gives us the sHD system (5).

Remark 1. In the pure bosonic case, we only have three cases which correspond to the KdV, MKdV and HD systems respectively [10, 11].

Remark 2. A simple calculation shows that the next one $L_t = [(L^{\frac{3}{2}})_{\geqslant 4}, L]$ will not lead to any consistent system. Thus, here we have only four cases.

For a general even-order differential operator

$$L = \sum_{i=0}^{n} u_i D^{2i} + \sum_{i=1}^{n} \alpha_i D^{2i-1} \tag{12}$$

we may consider the following Lax equation

$$L_t = [(L^{\frac{k}{n}})_{\geqslant r}, L] \qquad r = 0, 1, 2, 3. \tag{13}$$

The $r = 0, 1$ cases are considered in [12] and [6] respectively. It is pointed out that one may set the fields $u_n = \alpha_n = u_{n-1} = 0$ to zero in the case $r = 0$ and $u_n = \alpha_n = u_0 = 0$ in the case $r = 1$. When $r = 2$ and $r = 3$, all the field variables can be taken as dynamical variables. However, the following reductions or restrictions are feasible: $u_0 = 0, \alpha_1 = 0$ for $r = 2$ case and $u_1 = u_0 = 0, \alpha_1 = 0$ for the case $r = 3$.

Systems (13) are integrable in the sense that they consist of commuting flows. The proof of this statement is not difficult: the first two cases are proved in the cited references. The proof for the last two cases follows from the standard argument.

We conclude this letter with the following remarks.

(i) We see that for operator (12), we have the four cases mentioned above. This phenomenon is based on the following algebraic decompositions:

$$g = \left\{ \sum_i u_i D^i \right\} = g_{\geqslant r} \oplus g_{<r} \qquad r = 0, 1, 2, 3. \tag{14}$$

(ii) It would be interesting to study the Hamiltonian structures of our system sHD. We know that the HD equation is not only Hamiltonian but bi-Hamiltonian. We suspect that this is also the case for the sHD.

(iii) It has been proved that the cases $r = 0$ and $r = 1$ are gauge related to each other [6]. It is important to study the relationship between the $r = 1$, $r = 2$ and $r = 3$ cases.

(iv) We may construct the flows in terms of Sato's approach. The candidate for the pseudo-differential operator is $L = u_0 \partial + \alpha_0 D + \cdots$.

(v) In a recent paper [13], Darboux transformations for the sKdV are constructed. The same consideration will be interesting for the general cases.

I should like to thank Boris Kupershmidt for communications on the sHD equation. It is my pleasure to thank the referee for useful comments. This work is supported by the National Natural Science Foundation of China.

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